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# $\langle q, r\rangle$-number systems and algebraic independence

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This is an announcement of our results in [9].

let $q$ and $r$ are integers with $q \geq 2$ and $0 \leq r \leq q-1$. In the $\langle q, r\rangle$*number system*, every integer $n \in \mathbb{Z}$ is uniquely expressed with base $q$ and digits $-r, 1-r, \cdots, 0, \cdots, q-1-r$; namely,

$$n=\sum_{h=0}^{k} \delta_h q^h, \quad \delta_k \in \{-r, 1-r, \cdots, q-1-r\}, \quad \delta_k \neq 0 \text{ if } n \neq 0, \tag{1}$$

where $\mathbb{Z}$ should be replaced by $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{\leq 0}$ if $r=0$ and $r=q-1$, respectively. The usual $q$-adic expansion is the $\langle q, 0\rangle$ number system. Symmetrically, in the $\langle q, q-1-r\rangle$ number system $-n$ is uniquely expressed as

$$-n=\sum_{h=0}^{k}(-\delta_h) q^h, \tag{2}$$

where $\delta_h$ are as above (cf. [3], [5]).

Furthermore, taking the negative base $-q$, we have the $\langle -q, r\rangle$ number system, in which every $n \in \mathbb{Z}$ is uniquely expressed as

$$n=\sum_{h=0}^{l} \varepsilon_h(-q)^h, \quad \varepsilon_h \in \{-r, 1-r, \cdots, q-1-r\}, \ \varepsilon_l \neq 0 \text{ if } n \neq 0 \tag{3}$$

(without exception on $r$). In the $\langle -q, q-1-r\rangle$ number system, we have also an expansion of $-n$ similar to (2).

An arithmetical function $a_r(n): \mathbb{Z} \rightarrow \mathbb{C}$ is called $\langle q, r\rangle$-*linear*, if there is an $\alpha \in \mathbb{C}^{\times}$ such that

$$a_r(nq+t)=\alpha a_r(n)+a_r(t) \tag{4}$$

for any $n \in \mathbb{Z}$ and $t \in \mathbb{Z}$ with $-r \leq t \leq q-1-r$, where $\mathbb{Z}$ is replaced by $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{\leq 0}$ if $r=0$ and $r=q-1$, respectively. By definition, $a_r(0)=0$. Using the expansion (1), we have

$$a_r(n) = \sum_{h=0}^{k} a_r(\delta_h)\alpha^h, \tag{5}$$

and so $a_r(n)$ is determined by the *coefficient* $\alpha$ and the *initial vector*

$$\boldsymbol{a}_r = (a_r(-r), a_r(1-r), \dots, a_r(0), \dots, a_r(q-1-r)). \tag{6}$$

It follows from (2) and (5) that

$$a_{q-1-r}(-n) = \sum_{h=0}^{k} a_{q-1-r}(-\delta_h)\alpha^h. \tag{7}$$

An arithmetical function $b_r(n): \mathbb{Z} \to \mathbb{C}$ is called $\langle -q, r\rangle$*-linear*, if there is a $\beta \in \mathbb{C}^{\times}$ such that

$$b_r(n(-q)+t) = \beta b_r(n) + b_r(t) \tag{8}$$

for any $n \in \mathbb{Z}$ and $t \in \mathbb{Z}$ with $-r \leq t \leq q-1-r$. We have $b_r(0)=0$ and

$$b_r(n) = \sum_{h=0}^{l} b_r(\epsilon_h)\beta^h, \tag{9}$$

using the expression (3), so that $b_r(n)$ is determined by the coefficient $\beta$ and the initial vector

$$\boldsymbol{b}_r = (b_r(-r), b_r(1-r), \dots, b_r(0), \dots, b_r(q-1-r)).$$

For $b_{q-1-r}(n)$, we have an expression similar to (7)

**Examples.** We give some examples of $\langle q, r\rangle$-linear functions using the expression (1) of $n \in \mathbb{Z}$, where $\mathbb{Z}$ should be replaced by $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{\leq 0}$ if $r=0$ and $r=q-1$, respectively.

1. The *sum of digits function* in the $\langle q, r\rangle$ number system defined by $s_{\langle q,r\rangle}(n) = \sum_{h=0}^{k}\delta_h$ is $\langle q, r\rangle$-linear with the coefficient 1 and the initial vector $(-r, 1-r, \dots, q-1-r)$. Delange[1] proved for the ordinary $q$-adic sum of digits function $s_q(n) = s_{\langle q,0\rangle}(n)$ that

$$\frac{1}{N}\sum_{n<N} s_g(n) = \frac{q-1}{2}\log_q N + F(\log_q N), \tag{10}$$

where $F(x)$ is a continuous, nowhere differentiable function of period 1, whose Foureir coefficients are given explicitly. Flajolet and Ramshaw[3] and Grabner and Thuswaldner[4] studied these phenomena in the $\langle q,r\rangle$ number systems and in the $-q$ adic ones, respectively

2. For any given $t=-r,1-r,\cdots,q-1-r$, $e_{tr}(n)$ denotes the number of the digits $t$ appearing in the $\langle q,r\rangle$-expansion (1) of $n\in\mathbb{Z}$, which is $\langle q,r\rangle$-linear with the coefficient 1 and the initial conditions $e_{tr}(s)=1$ if $s=t$; $=0$ other wise. Flajolet and Ramshaw[3] proved Delange-type results for $e_{tr}(n)(-r\le t\le q-1-r)$ and applied them to the study of the summeatory functions of $s_{\langle q,r\rangle}=\sum_{t=-r}^{q-1-r}te_{tr}(n)$.

3. The *radical inverse function* in the $\langle q,r\rangle$ number system defined by $\phi_{\langle q,r\rangle}(n)=\sum_{h=0}^{k}\delta_h q^{-h-1}$ is $\langle q,r\rangle$-linear with the coefficient $q^{-1}$ and the initial vector $q^{-1}(-r,1-r,\ldots,q-1-r)$. Furthermore, for any given permutation $\sigma$ of $\{-r,1-r,\ldots,q-1-r\}$ with $0^\sigma=0$, the *generalized radical inverse function* defined by $\phi^\sigma_{\langle q,r\rangle}(n)=\sum_{h=0}^{k}\delta_h^\sigma q^{-h-1}$ is $\langle q,r\rangle$-linear with the coefficient $q^{-1}$ and the initial vector $q^{-1}((-r)^\sigma,(1-r)^\sigma,\ldots,(q-1-r)^\sigma)$ (cf. [8] Chapter 3).

4. For any given $p\in\mathbb{Z}$ with $|p|\ge q$, the bases change function $\gamma_{pqr}(n)$ is defined by $\gamma_{pqr}(n)=\sum_{h=0}^{k}\delta_h p^h$, which is $\langle q,r\rangle$-linear with the coefficient $p$ and the initial vector $(-r,1-r,\ldots,q-1-r)$ (cf. [2]).

5. The linear function $cn$ $(c\in\mathbb{C}^\times)$ is $\langle q,r\rangle$-linear with the coefficient $q$ and the initial vector $c(-r,1-r,\ldots,q-1-r)$.

Examples of $\langle -q,r\rangle$-linear functions can be constructed similarly as above by using the expression (3).

Recently, Kurosawa and the second named author[6] gave a necessarily and sufficient condition for the generating functions of $\langle q,0\rangle$-linear functions and $\langle -q,0\rangle$-linear ones to be algebraically independent over $\mathbb{C}(z)$. We note that the generating function of $a(n)=cn$ given in Example 5 is

$$\frac{z}{(1-z)^2}\in\mathbb{C}(z).$$

We state our theorems. Let $\alpha_i,\beta_i\in\mathbb{C}^\times$ $(1\le i\le I)$ satisfy

$$\alpha_i\ne\alpha_j,\ \beta_i\ne\beta_j\ (i\ne j,1\le i,j\le I). \tag{11}$$

For any fixed $q$, let $a_{ilr}(n)$ $(1\le l\le m(i))$ and $b_{ilr}(n)$ $(1\le l\le n(i))$ be $\langle q,r\rangle$-linear functions and $\langle -q,r\rangle$-linear ones with coefficients $\alpha_i$ and $\beta_i$, respectively. We consider the generating functions

$$f_{ilr}(z)=\sum_{n=0}^{\infty}a_{ilr}(n)z^n,\quad f^*_{ilr}(z)=\sum_{n=0}^{\infty}a_{ilr}(-n)z^n,$$

$$g_{ilr}(z)=\sum_{n=0}^{\infty} b_{ilr}(n)z^n, \quad g^*_{ilr}(z)=\sum_{n=0}^{\infty} b_{ilr}(-n)z^n,$$

which converge in $|z|<1$ by (4) and (8). We put

$$\boldsymbol{a}_{ilr}=(a_{ilr}(-r),a_{ilr}(1-r),\dots,a_{ilr}(q-1-r)),$$

$$\boldsymbol{b}_{ilr}=(b_{ilr}(-r),b_{ilr}(1-r),\dots,b_{ilr}(q-1-r)).$$

For any vector $\mathbf{c}=(c_1,c_2,\cdots,c_q)$, we write $\overleftarrow{\mathbf{c}}=(c_q,c_{q-1},\cdots,c_1)$.

**Theorem 1.1.** *The functions* $f_{ilr}(z)$ $(1\le i\le I, 1\le l\le m(i), 0\le r<q-1)$, $f^*_{ilr}(z)$ $(1\le i\le I, 1\le l\le m(i), 0<r\le q-1)$, $g_{ilr}(z)$ *and* $g^*_{ilr}(z)(1\le i\le I, 1\le l\le n(i), 0\le r\le q-1, 2r\ne q-1)$ *are algebraically independent over* $\mathbb{C}(z)$ *if and only if the following conditions (i) and (ii) hold;*

*(i) each one of the sets of vectors* $\{\boldsymbol{a}_{ilr},\overleftarrow{\boldsymbol{a}}_{ilq-1-r}; 1\le l\le m(i)\}$ $(1\le i\le I, 0\le r<q-1)$ *and* $\{\boldsymbol{b}_{ilr},\overleftarrow{\boldsymbol{b}}_{ilq-1-r}; 1\le l\le n(i)\}$ $(1\le i\le I, 0\le r\le q-1, 2r\ne q-1)$ *is linearly independent over* $\mathbb{C}$,

*(ii) if* $\alpha_i=q$, *then for any* $r$ *with* $0\le r<q-1$

$$(-r,1-r,\dots,q-1-r)\notin \mathrm{Span}_{\mathbb{C}}\{\boldsymbol{a}_{ilr},\overleftarrow{\boldsymbol{a}}_{ilq-1-r}; 1\le l\le m(i)\},$$

*and if* $\beta_i=-q$, *then for any* $r$ *with* $0\le r\le q-1$, $2r\ne q-1$

$$(-r,1-r,\dots,q-1-r)\notin \mathrm{Span}_{\mathbb{C}}\{\boldsymbol{b}_{ilr},\overleftarrow{\boldsymbol{b}}_{ilq-1-r}; 1\le j\le n(i)\}.$$

**Remark 1.1** To prove the theorem, we use a criterion of algebraic independence over $\mathbb{C}(z)$ of functions satisfying certain functional equations (cf. [7] Corollary of Theorem 3.2.1), which enable us to reduce the algebraic dependency over $\mathbb{C}(z)$ of our functions to the linear dependency of them over $\mathbb{C} \bmod \mathbb{C}(z)$. So we actually prove that the functions in the theorem are algebraically dependent over $\mathbb{C}(z)$ if and only if, for some $i$ and $r$, $f_{ilr}(z), f^*_{ilq-1-r}(z)$ $(1\le l\le m(i))$ are linearly dependent over $\mathbb{C}$, $g_{ilr}(z), g^*_{ilq-1-r}(z)$ $(1\le l\le n(i))$ are linearly dependent over $\mathbb{C}$, $\alpha_i=q$ and $z/(1-z)^2\in \mathrm{Span}_{\mathbb{C}}\{f_{ilr}(z),f^*_{ilq-1-r}(z); 1\le l\le m(i)\}$, or $\beta_i=-q$ and $z/(1-z)^2\in \mathrm{Span}_{\mathbb{C}}\{g_{ilr}(z),g^*_{ilq-1-r}(z); 1\le l\le n(i)\}$.

**Remark 1.2** The conditions (i) and (ii) in Theorem 1.1 imply that $m(i),n(i)\le q$ for any $i$, $\alpha_i\ne q$ if $m(i)=q$, and $\beta_i\ne -q$ if $n(i)=q$.

**Theorem 1.2.** *Let the functions* $f_{ilr}(z), f^*_{ilr}(z), g_{ilr}(z)$, *and* $g^*_{ilr}(z)$ *satisfy the conditions (i) and (ii) in Theorem1.1. Assume that* $\alpha_i,\beta_i,a_{ilr}(n)$, *and* $b_{ilr}(n)$ *are algebraic for all* $i,l,r$ *and* $n$. *Then, for any algebraic number* $\alpha$ *with* $0<|\alpha|<1$, *the numbers* $f_{ilr}(\alpha)$ $(1\le i\le I, 1\le l\le m(i), 0\le r<q-1)$, $f^*_{ilr}(\alpha)$ $(1\le i\le I, 1\le l\le m(i), 0<r\le q-1)$, $g_{ilr}(\alpha)$ *and* $g^*_{ilr}(\alpha)$ $(1\le i\le I, 1\le l\le n(i), 0\le r\le q-1, 2r\ne q-1)$ *are algebraically independent.*

If we fix $r = 0$ in Theorem 1.1 and Theorem 1.2, we have the results of Kurosawa and the second named author[6] mentioned above. In their proof, they used another criterion ([7] Theorem 3.5) of algebraic independence of functions over $\mathbb{C}(z)$.

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